

ON THE CHROMATIC NUMBER OF STRUCTURED CAYLEY GRAPHS

MOHAMMAD BARDESTANI AND KEIVAN MALLAHI-KARAI

ABSTRACT. In this paper, we will study the chromatic number of Cayley graphs of algebraic groups that arise from algebraic constructions. Using Lang-Weil bound and representation theory of finite simple groups of Lie type, we will establish lower bounds on the chromatic number of these graphs. This provides a lower bound for the chromatic number of Cayley graphs of the regular graphs associated to the ring of $n \times n$ matrices over finite fields. Using Weil's bound for Kloosterman sums we will also prove an analogous result for SL_2 over finite rings.

1. INTRODUCTION

Let G be a group and let S be a symmetric subset of G that is a set satisfying $S^{-1} = S$. The Cayley graph of G with respect to S , denoted by $\text{Cay}(G, S)$, is the graph whose vertex set is identified with G , and vertices $g_1, g_2 \in G$ are declared adjacent if and only if $g_1^{-1}g_2 \in S$. The question of estimating the chromatic number of random Cayley graphs of fixed finite groups has been of recent interest. Recall that the chromatic number of a graph \mathcal{G} , denoted by $\chi(\mathcal{G})$, is the least cardinal c such that the vertices of \mathcal{G} can be partitioned into c sets (called color classes) such that no color class contains an edge in \mathcal{G} .

Alon [2] considered the random Cayley graphs of arbitrary finite groups and established strong asymptotic almost sure lower bounds for their chromatic number. In this paper we will study the analogous problem, where the pair (G, S) arise from an algebraic construction, and can thus be viewed as highly structured. More precisely, let $\mathbf{G} \subseteq GL_n(\mathbb{C})$ be a Chevalley group. This is an algebraic group that is obtained by a simple complex Lie algebra [18]. It is well-known that \mathbf{G} is defined over \mathbb{Z} , that is, \mathbf{G} as an algebraic set is given by the set of simultaneous zeros of a finite set of polynomials with integer coefficients. The reader interested in a concrete example can consider the special case $\mathbf{G} = SL_n$. Let $\tilde{\mathbf{S}}$ be a Zariski closed subset of \mathbf{G} defined over \mathbb{Z} :

$$\tilde{\mathbf{S}} = \{(a_{ij}) \in \mathbf{G} : P_1(a_{ij}) = \cdots = P_r(a_{ij}) = 0\},$$

where P_1, \dots, P_r are polynomials with integer coefficients. Since \mathbf{G} and $\tilde{\mathbf{S}}$ are defined over \mathbb{Z} , we can consider the \mathbb{F}_q -points of \mathbf{G} and $\tilde{\mathbf{S}}$ denoted by $\mathbf{G}(\mathbb{F}_q)$ and $\tilde{\mathbf{S}}(\mathbb{F}_q)$. Assume that $\mathbf{1} \notin \tilde{\mathbf{S}}$ where $\mathbf{1}$ is the identity element of \mathbf{G} and denote by

$$(1) \quad \mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q) := \text{Cay}(\mathbf{G}(\mathbb{F}_q), \mathbf{S}(\mathbb{F}_q)),$$

the Cayley graph of the group $\mathbf{G}(\mathbb{F}_q)$ with respect to the symmetrized set $\mathbf{S}(\mathbb{F}_q) = \tilde{\mathbf{S}}(\mathbb{F}_q) \cup \tilde{\mathbf{S}}(\mathbb{F}_q)^{-1}$.

A special case of this construction is the *regular graph*

$$RG_n(\mathbb{F}_q) = \text{Cay}(SL_n(\mathbb{F}_q), \mathbf{S}(\mathbb{F}_q)),$$

where $\mathbf{S} \subseteq SL_n$ is defined by a single polynomial $P(x_{ij}) = \det(I + (x_{ij}))$ and I is the identity matrix. In other words the regular graph $RG_n(\mathbb{F}_q)$ is a graph with the vertex set $SL_n(\mathbb{F}_q)$, in which two matrices x, y form an edge if and only if $\det(x + y) = 0$. The concept of regular graph was introduced by Anderson and Badawi [3] and it has been the source of several investigations. Akbari, Jamaali and Fakhari [1] showed that the clique number of $RG_n(\mathbb{F}_q)$ is bounded by a universal constant independent of \mathbb{F}_q (as long as q is odd). Moreover Tomon [19] showed that the chromatic number of $RG_n(\mathbb{F}_q)$ is at least $(q/4)^{\lfloor n/2 \rfloor}$, when $q = p^f$ is odd.

One can pose various questions: for instance, is it true that as $q \rightarrow \infty$, we have $\chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q)) \rightarrow \infty$, where $\chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q))$ is the chromatic number of the graph $\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q)$. Once the qualitative statement is established, one can also inquire about the growth of $\chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q))$ as a function of q . Using Lang-Weil bound and representation theory of simple groups of Lie type, we prove the following result which answers a far reaching generalization of question 525 proposed in the 22nd British Combinatorics Conference 2009 [8].

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Theorem 1.1. *Let $\mathbf{G} \subseteq \mathrm{GL}_n(\mathbb{C})$ be a simple and simply-connected Chevalley group of rank $r = \mathrm{rank} \mathbf{G}$ and dimension $d = \dim \mathbf{G}$, and let $\tilde{\mathbf{S}}$ be a geometrically irreducible Zariski closed subset of \mathbf{G} , defined over \mathbb{Z} , with codimension $m = \mathrm{codim} \mathbf{S}$. Then there exists a constant $C = C(d, r, m)$ such that for all but finitely many primes $p \geq 3$*

$$(2) \quad \chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q)) \geq Cq^{\frac{r-m}{2}},$$

where the characteristic of \mathbb{F}_q is p and $\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q)$ is the graph defined by (1).

There are many spectral bounds for the chromatic number of Cayley graph. These methods are effective when the character theory of the underlying group is easily understood (for instance, when it is abelian), but they become prohibitively hard for more complicated groups. Theorem 1.1 provides a decent lower bound for the chromatic number by exploiting the quasirandomness of the underlying groups. One can thus view this theorem as a soft spectral bound. Following Gowers [14], a finite group G is called D -quasirandom if every non-trivial complex representation of G has dimension at least D . Frobenius first observe that $\mathrm{SL}_2(\mathbb{F}_p)$ is $(p-1)/2$ -quasirandom. This result has been extended to all finite group of Lie type by Landazuri and Seitz [15].

Remark 1.2. It would be interesting to see how sharp the bound (2) is. In particular, we do not know what happens when $\mathrm{rank} \mathbf{G} \leq \mathrm{codim} \mathbf{S}$. Here is an interesting test case. Fix $n \geq 2$ and $0 \leq \ell \leq n$. Define the graph $\mathrm{RG}_{n, \ell}(\mathbb{F}_q)$ with the vertex set $\mathrm{SL}_n(\mathbb{F}_q)$, in which two matrices x, y form an edge if and only if $\mathrm{rank}(x + y) \leq n - \ell$. For $\ell = 1$, we recover the regular graphs $\mathrm{RG}_n(\mathbb{F}_q)$. Notice that all the connected components of the algebraic variety $\{x \in \mathrm{SL}_n(\mathbb{C}) : \mathrm{rank}(I + x) \leq n - \ell\}$ have codimension ℓ^2 and so by Theorem 1.1 we obtain

$$(3) \quad \chi(\mathrm{RG}_{n, \ell}(\mathbb{F}_q)) \gg_{n, \ell} q^{\frac{n-\ell^2-1}{2}}.$$

In particular from (3) we obtain

$$\chi(\mathrm{RG}_n(\mathbb{F}_q)) \gg q^{(n-2)/2}.$$

This bound is non-trivial as long as $\ell \ll \sqrt{n}$, but we do not know what happens even for $\ell = n - 1$.

Let us explain the reason for considering simple algebraic groups by showing an obvious obstruction to the growth of the chromatic numbers with q .

Example 1.3 (Abelian quotients as an obstruction). Let \mathbf{A} be either \mathbb{G}_a or \mathbb{G}_m and let \mathbf{S} be a proper Zariski closed subset of \mathbf{A} defined over \mathbb{Z} , and assume that $1 \notin \mathbf{S}$. For simplicity, assume that $\mathbf{S} = \{\pm a\}$ in the case of the additive group, and $\mathbf{S} = \{a^{\pm 1}\}$ in the case of the multiplicative group. It is easy to see that (except for a finite number of characteristics) for any q , each connected component of the graph $\mathcal{G}_{\mathbf{A}, \mathbf{S}}(\mathbb{F}_q)$ is isomorphic to a cycle and so $\chi(\mathcal{G}_{\mathbf{A}, \mathbf{S}}(\mathbb{F}_q)) \leq 3$. More generally, let $\phi : \mathbf{G} \rightarrow \mathbf{A}$ be a nontrivial morphism of algebraic groups defined over \mathbb{Z} . Set $\mathbf{S}_1 = \phi^{-1}(\mathbf{S})$. We easily see that $\chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}_1}(\mathbb{F}_q)) \leq \chi(\mathcal{G}_{\mathbf{A}, \mathbf{S}}(\mathbb{F}_q))$ and hence its chromatic number is also bounded by 3. Note also that $\mathbf{S}_1 = \phi^{-1}(\mathbf{S})$ has codimension one in \mathbf{G} . This example shows that Theorem 1.1 is not unconditionally true for all algebraic groups. It would be interesting to know if other obstructions exist.

As an immediate corollary of Theorem 1.1 we have

Corollary 1.4. *Let \mathbf{G} and $\tilde{\mathbf{S}}$ be as in Theorem 1.1 and assume that $\mathrm{rank} \mathbf{G} > \mathrm{codim} \tilde{\mathbf{S}}$. Then for all but finitely many primes $p \geq 3$*

$$\chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\overline{\mathbb{F}_p})) = \infty,$$

where $\overline{\mathbb{F}_p}$ is the algebraic closure of \mathbb{F}_p and $p \geq 3$.

It is noteworthy that the exponential growth of the chromatic number is not due to the existence of a large clique. In fact, assuming $1 \notin \tilde{\mathbf{S}}$, one can easily show that there exists a constant $C = C_{\tilde{\mathbf{S}}}$ such that for all except finite number of characteristics, we have $\omega(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q)) \leq C$, where $\omega(\mathcal{G})$ denotes the clique number of a graph \mathcal{G} .

As pointed out above, our method does not give any non-trivial bound when $\mathrm{rank} \mathbf{G} \leq \mathrm{codim} \tilde{\mathbf{S}}$. In some cases, one can invoke spectral bounds to give relatively sharp estimates. Our next theorem is a special case. Recall that the regular graph $\mathrm{RG}_2(\mathbb{F}_q)$ is a graph with the vertex set $\mathrm{SL}_2(\mathbb{F}_q)$, in which two matrices x, y form an edge if and only if $\det(x + y) = 0$.

Theorem 1.5. *Let \mathbb{F}_q be a finite field of cardinality q , and characteristic $p \geq 3$. Then we have*

$$(4) \quad q + 1 \leq \chi(\mathrm{RG}_2(\mathbb{F}_q)) \leq 8(q + 1).$$

Remark 1.6. Previously Tomon [19] has considered the graph $\Gamma_2(\mathbb{F}_q)$ with the vertex set $\text{GL}_2(\mathbb{F}_q)$, in which two vertices $x, y \in \text{GL}_2(\mathbb{F}_q)$ form an edge if $\det(x + y) = 0$. He showed that for all prime powers q , the inequalities $q/4 \leq \chi(\Gamma_2(\mathbb{F}_q)) \leq 4q(q + 1)$ hold. The method of [19], however, does not seem to be applicable to the subgraph $\text{RG}_2(\mathbb{F}_q)$, and will only yield a trivial bound. Moreover by modifying the proof of Theorem 1.5 one can show that $\chi(\Gamma_2(\mathbb{F}_p)) \ll p$ when $p \equiv 3 \pmod{4}$. Therefore one might expect that the correct order of $\chi(\Gamma_2(\mathbb{F}_q))$ should be q rather than q^2 .

Remark 1.7. When -1 is a quadratic non-residue in the finite field \mathbb{F}_q , the upper bound in (4) can be improved to $2(q + 1)$. It is clear from Theorem 1.5 that $\chi(\text{RG}_2(\mathbb{F}_q))$ grows linearly with q . It would be interesting to obtain sharper lower and upper bounds. For instance, one can ask if $\lim_p \chi(\text{RG}_2(\mathbb{F}_p))/p$ exists. Similarly, (4) may suggest that $\chi(\text{RG}_n(\mathbb{F}_q))$ has the order q^{n-1} .

The regular graph $\text{RG}_n(\mathbb{F}_q)$ can also be defined over finite rings. $\text{RG}_n(\mathbb{Z}/p^r\mathbb{Z})$ is a graph with the vertex set $\text{SL}_n(\mathbb{Z}/p^r\mathbb{Z})$, in which two matrices x, y form an edge if and only if $\det(x + y) = 0$. We now discuss lower bound for $\chi(\text{RG}_2(\mathbb{Z}/p^r\mathbb{Z}))$. Although it is conceivable that the character theory of $\text{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$ can be employed for this purpose, lengthy computations are bound to arise. We will circumvent the complications by finding certain simpler Cayley graphs inside $\text{RG}_2(\mathbb{Z}/p^r\mathbb{Z})$. The spectral bounds for the chromatic number of these graphs turn out to be intimately related to certain Kloosterman sums, for which classical estimates exist. Using this strategy we can prove the following theorem.

Theorem 1.8. *Let $p \geq 5$ be a prime number. Then*

$$\frac{\sqrt{p}}{4} \leq \chi(\text{RG}_2(\mathbb{Z}/p^r\mathbb{Z})) \leq 8(p + 1).$$

This paper is organized as follows: in Section 2 we will give a proof of Theorem 1.1. Sections 3 and 4 are devoted to the proof of Theorems 1.5 and 1.8.

2. QUASIRANDOM GROUPS

Quasirandom groups were introduced by Gowers [14] in order to answer a question of Babai and Sós on product-free sets in finite groups. We recall that a finite group G is called D -quasirandom if every non-trivial complex representation of G has dimension at least D . One of the main results in [14] is the following mixing inequality:

Theorem 2.1. *Let G be finite D -quasirandom group. If $A, B, C \subseteq G$ such that*

$$|A||B||C| > |G|^3/D,$$

then the set $AB \cap C$ is non-empty.

Gowers' proof, as well as the proof given later by Babai, Nikolov and Pyber [5], is based on spectral analysis of graphs. A Fourier analytic proof of this theorem can also be found in [7].

Lemma 2.2. *Let G be finite D -quasirandom group and let S be a symmetric subset of G with the associated Cayley graph $\text{Cay}(G, S)$. Assume that $\mathbf{1} \notin S$ where $\mathbf{1}$ is the identity element of G , then*

$$(5) \quad \chi(\text{Cay}(G, S)) \geq \sqrt{\frac{D|S|}{|G|}}.$$

Proof. let κ be the chromatic number of $\text{Cay}(G, S)$. Hence G can be partitioned into κ subsets A_1, \dots, A_κ such that $xy^{-1} \notin S$ when $x, y \in A_i$ for all $1 \leq i \leq \kappa$. There exists $1 \leq i \leq \kappa$ such that A_i has size at least $|G|/\kappa$. Set $A = A_i$, $B = A_i^{-1}$, and $C = S$. From the definition of chromatic number, we have $AB \cap S = \emptyset$. Therefore, by Theorem 2.1, we conclude

$$\frac{|G|^2|S|}{\kappa^2} \leq |A||B||S| = |A|^2|S| \leq \frac{|G|^3}{D},$$

which completes the proof. \square

Next we prove the following simple lemma which will be useful later. For a given group G , the minimal dimension of non-trivial irreducible representations of G is denoted by $m(G)$ and $m_{\text{proj}}(G)$ denotes the minimal dimension of non-trivial irreducible projective representations of G .

Lemma 2.3. *Let G be a perfect group. Then $m(G) \geq m_{\text{proj}}(G/Z(G))$, where $Z(G)$ is the center of G .*

Proof. Let $\rho : G \rightarrow \mathrm{GL}_n(\mathbb{C})$ be a non-trivial irreducible representation. Since G is perfect, and thus does not have any non-trivial one dimensional representation, we obtain a non-trivial projective representation $\bar{\rho} : G \rightarrow \mathrm{PGL}_n(\mathbb{C})$. Moreover from irreducibility of ρ we conclude that $Z(G) \subseteq \ker \bar{\rho}$ and so we obtain a non-trivial reducible projective representation $\bar{\rho} : G/Z(G) \rightarrow \mathrm{PGL}_n(\mathbb{C})$ and so $n \geq m_{\mathrm{proj}}(G/(Z(G)))$. \square

Proof of Theorem 1.1. Let $d = \dim \mathbf{G}$, $r = \mathrm{rank} \mathbf{G}$, $m = \mathrm{codim} \tilde{\mathbf{S}}$ and $q = p^f$. By applying Schwarz-Zippel bound [16, Lemma 1] we observe that $|\mathbf{G}(\mathbb{F}_q)| \leq c_1 q^d$ where c_1 is independent of q . Since $\tilde{\mathbf{S}}$ is geometrically irreducible, then by Bertini-Noether [13, Corollary 10.4.3] we can conclude that, for all but finitely many primes p , the variety $\tilde{\mathbf{S}}(\mathbb{F}_q)$ is geometrically irreducible with the same dimension as $\tilde{\mathbf{S}}$. Hence by Lang-Weil bound [16], we obtain $|\mathbf{S}(\mathbb{F}_q)| \geq c_2 q^{d-m}$, where c_2 is independent of q and $\mathbf{S}(\mathbb{F}_q) = \tilde{\mathbf{S}}(\mathbb{F}_q) \cup \tilde{\mathbf{S}}(\mathbb{F}_q)^{-1}$. Therefore

$$\frac{|\mathbf{S}(\mathbb{F}_q)|}{|\mathbf{G}(\mathbb{F}_q)|} \geq \frac{c_3}{q^m},$$

where c_3 is independent of q . Moreover, for all but finitely many primes p , the finite group $\mathbf{G}(\mathbb{F}_q)$ is perfect and $\mathbf{G}(\mathbb{F}_q)/Z(\mathbf{G}(\mathbb{F}_q))$ is simple [17, Theorem 24.17]. By Landazuri and Seitz theorem [15], we have $m_{\mathrm{proj}}(\mathbf{G}(\mathbb{F}_q)/Z(\mathbf{G}(\mathbb{F}_q))) \geq c_4 q^{\mathrm{rank}(\mathbf{G})}$, where c_4 is independent of q and so by Lemma 2.3 we can conclude that $\mathbf{G}(\mathbb{F}_q)$ is $O(q^r)$ -quasirandom. Since $\mathbf{1} \notin \tilde{\mathbf{S}}$, then by applying Lemma 2.2 we have

$$\chi(\mathcal{G}_{\mathbf{G}, \mathbf{S}}(\mathbb{F}_q)) \gg q^{\frac{r-m}{2}}.$$

\square

3. CAYLEY GRAPHS AND THEIR SPECTRA

We first recall some well-known facts from algebraic graph theory. The following spectral bound for the chromatic number of graphs which is due to Hoffman and we refer the reader to [6, Theorem 7, page 265] for its proof.

Lemma 3.1. *Let \mathcal{G} be a non-empty graph with n vertices. Then*

$$(6) \quad \chi(\mathcal{G}) \geq 1 - \lambda_0 / \lambda_{n-1}.$$

where $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$ is the spectrum of the adjacency matrix of \mathcal{G} .

In some cases it will be more convenient to use the following lower bound [9, 1.5.4. Corollary].

Lemma 3.2. *Let \mathcal{G} be a finite, connected, ℓ -regular graph on n vertices, without loop. Then*

$$\chi(\mathcal{G}) \geq \frac{\ell}{\max\{|\lambda_1|, |\lambda_{n-1}|\}}.$$

Now let G be a group and let $S \subseteq G$ be a symmetric set. Note that the Cayley graph $\mathrm{Cay}(G, S)$ is $|S|$ -regular and is connected if and only if S generates G . The following theorem due to Babai [4] and Diaconis-Shahshahani [10], describes the spectrum of the adjacency matrix of $\mathrm{Cay}(G, S)$ using the character theory of G .

Theorem 3.3. *Let G be a finite group and S a symmetric subset which is stable under conjugation. Let A be the adjacency matrix of the graph $\mathrm{Cay}(G, S)$. Then the eigenvalues of A are given by*

$$\lambda_\rho = \frac{1}{\dim(\rho)} \sum_{s \in S} \chi_\rho(s),$$

as χ_ρ , the character of the representation ρ , ranges over all irreducible characters of G . Moreover, the multiplicity of λ_ρ is $\dim(\rho)^2$.

We now turn to the graph $\mathrm{RG}_2(\mathbb{F}_q)$, $q = p^f$ for an odd prime p . We remark that the method of the previous section is not applicable to $\mathrm{RG}_2(\mathbb{F}_q)$. Instead we use representation theory to obtain a lower bound for the chromatic number of this graph. As before, $E_{2,q}$ denotes the set of matrices in $\mathrm{SL}_2(\mathbb{F}_q)$ which have -1 as an eigenvalue. Since $E_{2,q}$ is a union of conjugacy classes, we can use the Jordan canonical form to give a simple description of the set $E_{2,q}$. Let ν be a generator of the cyclic group \mathbb{F}_q^* . Any matrix in $\mathrm{SL}_2(\mathbb{F}_q)$ with an eigenvalue -1 is either $-I$, where I is the identity matrix, or is conjugate in $\mathrm{SL}_2(\mathbb{F}_q)$ to one of the following matrices

$$T_1 := \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad T_2 := \begin{pmatrix} -1 & 0 \\ -\nu & -1 \end{pmatrix}.$$

Hence $E_{2,q} = \{-I, (T_1), (T_2)\}$, where (T_1) and (T_2) denote the conjugacy classes of T_1 and T_2 . It is easy to see that each of these conjugacy classes has $(q^2 - 1)/2$ elements and so $|E_{2,q}| = q^2$. We recall that $\text{SL}_2(\mathbb{F}_q)$ is generated by unipotent matrices. Notice that the subgroup generated by $E_{2,q}$ contains

$$(7) \quad \begin{pmatrix} -1 & 0 \\ a & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{F}_q.$$

Hence $E_{2,q}$ generate $\text{SL}_2(\mathbb{F}_q)$ from which it follows that the graph $\text{RG}_2(\mathbb{F}_q) = \text{Cay}(\text{SL}_2(\mathbb{F}_q), E_{2,q})$ is a q^2 -regular connected graph. Let A be the adjacency matrix of $\text{RG}_2(\mathbb{F}_q)$ with eigenvalues

$$\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}, \quad n = |\text{SL}_2(\mathbb{F}_q)|.$$

Since $\text{RG}_2(\mathbb{F}_q)$ is a q^2 -regular connected graph then we have

$$(8) \quad q^2 = \lambda_0 > \lambda_1 \geq 0 > \lambda_{n-1}.$$

We now invoke Lemma 3.1 to find a lower bound of $\chi(\text{RG}_2(\mathbb{F}_q))$. In order to apply this lemma we need to estimate the size of the eigenvalues of the adjacency matrix, which by Theorem 3.3 are given by

$$(9) \quad \lambda_\rho = \frac{1}{\dim(\rho)} \sum_{s \in E_{2,q}} \chi_\rho(s) = \frac{1}{\dim(\rho)} \left(\chi_\rho(-I) + \frac{q^2 - 1}{2} \chi_\rho(T_1) + \frac{q^2 - 1}{2} \chi_\rho(T_2) \right),$$

where χ_ρ , the character of the representation ρ , ranges over all irreducible characters of $\text{SL}_2(\mathbb{F}_q)$. Representations of $\text{SL}_2(\mathbb{F}_q)$ have been studied by Frobenius and Schur. For more details we refer the reader to [11, Section 38]. From (8) and (9), to evaluate λ_{n-1} we only need to know the values of non-trivial characters at $-I, T_1$ and T_2 .

Denote $\varepsilon = (-1)^{(q-1)/2}$. Then for $1 \leq i \leq (q-3)/2$ and $1 \leq j \leq (q-1)/2$ we have the following table:

Rep	dim	$-I$	T_1	T_2
ψ	q	q	0	0
χ_i	$q+1$	$(-1)^i(q+1)$	$(-1)^i$	$(-1)^i$
θ_j	$q-1$	$(-1)^j(q-1)$	$(-1)^{j+1}$	$(-1)^{j+1}$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}\varepsilon(1 + \sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1 - \sqrt{\varepsilon q})$
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}\varepsilon(1 - \sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1 + \sqrt{\varepsilon q})$
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}\varepsilon(1 - \sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1 + \sqrt{\varepsilon q})$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}\varepsilon(1 + \sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1 - \sqrt{\varepsilon q})$

From (9), the above table and a simple calculation we obtain the following equalities:

$$(10) \quad \lambda_\psi = 1, \quad \lambda_{\chi_i} = (-1)^i q, \quad \lambda_{\theta_j} = (-1)^{j+1} q, \quad \lambda_{\xi_1} = \lambda_{\xi_2} = \lambda_{\eta_1} = \lambda_{\eta_2} = \varepsilon q.$$

With these preliminaries, we are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Using (10) along with Theorem 3.3 we have:

$$\frac{\lambda_0}{-\lambda_{n-1}} = \frac{\lambda_0}{|\lambda_{n-1}|} = q.$$

By combining this with Lemma 3.1 we obtain the lower bound $q+1 \leq \chi(\text{RG}_2(\mathbb{F}_q))$. In order to establish the upper bound, we will exhibit a proper coloring of $\chi(\text{RG}_2(\mathbb{F}_q))$ with $8(q+1)$ colors. Let $\Sigma = \{0, 1, -1\}$ and $\lambda : \mathbb{F}_q \rightarrow \Sigma$ be a function satisfying $\lambda(0) = 0$ and $\lambda(x) \neq \lambda(-x)$ for all $x \in \mathbb{F}_q \setminus \{0\}$. Denote by B the subgroup of upper-triangular matrices in $G := \text{SL}_2(\mathbb{F}_q)$, and let $\pi : G \rightarrow G/B$ be the canonical map. Define the coloring map Θ by

$$\Theta : G \rightarrow G/B \times \Sigma \times \Sigma, \quad X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (\pi(X), \lambda(c), \lambda(d)).$$

We claim that Θ provides a proper coloring for $\text{RG}_2(\mathbb{F}_q)$. Let $X, Y \in \text{SL}_2(\mathbb{F}_q)$ be such that $\Theta(X) = \Theta(Y)$ and -1 is an eigenvalue of $X^{-1}Y$. From $\pi(X) = \pi(Y)$ we conclude that $X^{-1}Y \in B$. It is easy to see

that every element in B with an eigenvalue equal to -1 is of the form $\begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}$ for some $t \in \mathbb{F}_q$. Write $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -a & at - b \\ -c & ct - d \end{pmatrix}.$$

Since $\Theta(X) = \Theta(Y)$, we have $\lambda(c) = \lambda(-c)$, which implies that $c = 0$. Now, we must also have $\lambda(d) = \lambda(ct - d) = \lambda(-d)$, which implies that $d = 0$, which is a contradiction. So Θ provides a proper coloring and then (noting that $\lambda(c) = \lambda(d) = 0$ cannot occur), we obtain $\chi(\text{RG}_2(\mathbb{F}_q)) \leq 8|G/B| = 8(q+1)$. \square

When -1 is a quadratic non-residue in \mathbb{F}_q one can improve the upper bound to $2(q+1)$. Indeed let $H = (\mathbb{F}_q^*)^2$ be the quadratic residue subgroup which does not contain -1 . Now consider the following subgroup of $G := \text{SL}_2(\mathbb{F}_q)$:

$$B' = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x \in H, y \in \mathbb{F}_q \right\}.$$

Let $\pi : G \rightarrow G/B'$ to be canonical quotient map. We claim that π provides a proper coloring. If $\pi(X) = \pi(Y)$, then $X^{-1}Y \in B'$, which implies that the eigenvalues of $X^{-1}Y$ are distinct from -1 . Hence $\chi(\Gamma_2(\mathbb{F}_q)) \leq |G/B'| = 2(q+1)$. In a similar fashion, one obtains the upper bound $4(q+1)$ if -1 is not a fourth power in \mathbb{F}_q^* .

4. REGULAR GRAPHS OVER RINGS

This section is devoted to the proof of Theorem 1.8. Let R be a commutative ring with 1. The hyperbola graph over R , denoted by $\text{HG}(R)$, is defined by

$$\text{HG}(R) = \text{Cay}(R^2, \mathcal{S}), \quad \mathcal{S} = \{(x, y) \in R \times R : xy = 1\}.$$

Proposition 4.1. *Let R be a commutative ring in which 2 is invertible. Then there exists a subset $A \subseteq \text{SL}_2(R)$ of vertices of $\text{RG}_2(R)$ such that the induced subgraph on A is isomorphic to $\text{HG}(R)$.*

Proof. For $x, y \in R$ define

$$(11) \quad a_{x,y} := \begin{pmatrix} 1 & -2x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2y & 1 \end{pmatrix} = \begin{pmatrix} 1 - 4xy & -2x \\ 2y & 1 \end{pmatrix} \in \text{SL}_2(R).$$

A simple computation shows that $\det(a_{x_1,y_1} + a_{x_2,y_2}) = 4 - 4(x_2 - x_1)(y_2 - y_1)$. Hence, vertices a_{x_1,y_1} and a_{x_2,y_2} of $\text{RG}_2(R)$ are adjacent if and only if $(x_2 - x_1)(y_2 - y_1) = 1$. Note also that since 2 is invertible, the map $(x, y) \mapsto a_{x,y}$ is a injective. This implies that the set $A = \{a_{x,y} : x, y \in R\}$ fulfills the requirements. \square

Corollary 4.2. *For a commutative ring R in which 2 is invertible, we have $\chi(\text{RG}_2(R)) \geq \chi(\text{HG}(R))$.*

The upper bound is rather straightforward; the proof of the lower bound, however, relies on Estermann-Weil bounds for the Kloosterman sums.

Lemma 4.3. *Let $n \geq 2$, and let R be a ring with a proper ideal \mathfrak{a} such that 2 is not a zero divisor in R/\mathfrak{a} . Then $\chi(\text{RG}_n(R)) \leq \chi(\text{RG}_n(R/\mathfrak{a}))$ and $\chi(\text{HG}(R)) \leq \chi(\text{HG}(R/\mathfrak{a}))$.*

Proof. Let $\pi : \text{SL}_n(R) \rightarrow \text{SL}_n(R/\mathfrak{a})$ be the group homomorphism induced by the natural ring homomorphism $R \rightarrow R/\mathfrak{a}$. Let $V(\text{RG}_n(R/\mathfrak{a}))$ denotes the vertex set of the graph and consider a proper coloring $\Theta : V(\text{RG}_n(R/\mathfrak{a})) \rightarrow [k]$, and define $\tilde{\Theta} = \Theta \circ \pi : V(\text{RG}_n(R)) \rightarrow [k]$. We claim that $\tilde{\Theta}$ is a proper coloring of the graph $\text{RG}_n(R)$. To see this, assume that A and B form an edge in $\text{RG}_n(R)$. Then $\det(A + B) = 0$, implying that $\det(\pi(A) + \pi(B)) = 0$. If $\pi(A) \neq \pi(B)$, then we are done. If not, we have $2^n \det(\pi(A)) = 0$, which implies that $\det(\pi(A)) = 0$, contradicting the assumption that \mathfrak{a} is proper. The first inequality follows from here. the second inequality can be proven in a similar way. \square

We will also need the following straightforward facts about $\text{HG}(\mathbb{Z}/p^n\mathbb{Z})$.

Lemma 4.4. *Let $p \geq 5$ be a prime number and $n \geq 1$ a positive integer. Then the Cayley graph $\text{HG}(\mathbb{Z}/p^n\mathbb{Z})$ is a non-bipartite, connected $(p^n - p^{n-1})$ -regular graph.*

Proof. Obviously $\text{HG}(\mathbb{Z}/p^n\mathbb{Z}) = \text{Cay}(\mathbb{Z}/p^n\mathbb{Z}, \mathcal{S})$ is $(p^n - p^{n-1})$ -regular graph since

$$\mathcal{S} = \{(x, y) \in (\mathbb{Z}/p^n\mathbb{Z})^2 : xy = 1\},$$

has $(p^n - p^{n-1})$ elements. To show that $\text{HG}(\mathbb{Z}/p^n\mathbb{Z})$ is connected, we prove that \mathcal{S} generates the additive group $(\mathbb{Z}/p^n\mathbb{Z})^2$. Clearly $v_1 = (1, 1), v_2 = (2, 1/2) \in \mathcal{S}$ and $\det \begin{pmatrix} 1 & 1 \\ 2 & 1/2 \end{pmatrix} = -3/2$, which is a unit if $p \geq 5$. Hence $\{v_1, v_2\}$ generates $(\mathbb{Z}/p^n\mathbb{Z})^2$. Finally notice that for $0 \leq i \leq p^n - 1$, the vertices (i, i) and $(i+1, i+1)$ are adjacent. Since p is an odd prime $\text{HG}(\mathbb{Z}/p^n\mathbb{Z})$ contains an odd cycle and so $\text{HG}(\mathbb{Z}/p^n\mathbb{Z})$ is not a bipartite graph. \square

Let u, v be two integers and m a positive integer. The associated Kloosterman sum is defined by

$$\text{Kl}(u, v, m) := \sum_{\substack{x=1 \\ \gcd(x, m)=1}}^m \exp\left(\frac{2\pi i(ux + vx^*)}{m}\right),$$

where x^* is the inverse of x modulo m . By the Estermann-Weil bound [12], for $p \geq 3$ we have

$$(12) \quad |\text{Kl}(u, v, p^n)| \leq 2 \gcd(u, v, p^n)^{1/2} p^{n/2},$$

Proof of Theorem 1.8. Let $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{p^{2n}-1}$ be the spectrum of $\text{HG}(\mathbb{Z}/p^n\mathbb{Z})$. From Lemma 4.4 we have $\lambda_0 = p^n - p^{n-1}$ and

$$(13) \quad \max\{|\lambda_1|, |\lambda_{p^{2n}-1}|\} < p^n - p^{n-1}.$$

Since the $(\mathbb{Z}/p^n\mathbb{Z})^2$ is an abelian group then all of its irreducible representations are one-dimensional. So, by Theorem 3.3, for each $0 \leq i \leq p^{2n} - 1$ there exists two integers $1 \leq u_i, v_i \leq p^n$ such that

$$\lambda_i = \sum_{\substack{(x, y) \in (\mathbb{Z}/p^n\mathbb{Z})^2 \\ xy=1}} \exp\left(\frac{2\pi i(u_i x + v_i y)}{p^n}\right) = \text{Kl}(u_i, v_i, p^n).$$

By (13) for $1 \leq i \leq p^{2n} - 1$ we have $\gcd(u_i, v_i, p^n) \leq p^{n-1}$. From (12) we have

$$|\lambda_i| \leq 2p^{(n-1)/2} p^{n/2} = 2p^{n-1/2}, \quad 1 \leq i \leq p^{2n} - 1.$$

This implies that $\max\{|\lambda_1|, |\lambda_{p^{2n}-1}|\} \leq 2p^{n-1/2}$. Therefore by Lemma 3.2 we can deduce that

$$\chi(\text{HG}(\mathbb{Z}/p^n\mathbb{Z})) \geq \frac{p^n - p^{n-1}}{\max\{|\lambda_1|, |\lambda_{p^{2n}-1}|\}} \geq \frac{p^n - p^{n-1}}{2p^{n-1/2}} \geq \frac{\sqrt{p}}{4}.$$

Now, from Corollary 4.2, we have $\chi(\text{RG}_2(\mathbb{Z}/p^n\mathbb{Z})) \geq \chi(\text{HG}(\mathbb{Z}/p^n\mathbb{Z})) \geq \sqrt{p}/4$. The upper bound immediately follow from Lemma 4.3 for $\mathfrak{a} = p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, 585 KING EDWARD, OTTAWA, ON K1N 6N5, CANADA.
E-mail address: mbardest@uottawa.ca

JACOBS UNIVERSITY BREMEN, CAMPUS RING I, 28759 BREMEN, GERMANY.
E-mail address: k.mallahikarai@jacobs-university.de